ON A GENERALIZATION OF THE JORDAN CANONICAL FORM THEOREM ON SEPARABLE HILBERT SPACES

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ABSTRACT. We prove a generalization of the Jordan canonical form theorem for a class of bounded linear operators on complex separable Hilbert spaces.

1. INTRODUCTION

Throughout this paper, all Hilbert spaces discussed are *complex and separable* while all operators are bounded and linear on the Hilbert spaces. Let $\mathscr{H}$ be a Hilbert space and let $\mathscr{L}(\mathscr{H})$ be the set of bounded linear operators on $\mathscr{H}$. An *idempotent P* is an operator such that $P^2 = P$. A *projection Q* is an idempotent such that $Q = Q^*$. Unless otherwise stated, the term *algebra* will always refer to a unital subalgebra of $\mathscr{L}(\mathscr{H})$ which is closed in the strong operator topology. An operator A on $\mathscr{H}$ is said to be *irreducible* if its commutant $\{A\}' \equiv \{B \in \mathscr{L}(\mathscr{H}) : AB = BA\}$ contains no projections other than 0 and I, introduced by P. Halmos [7]. (The separability assumption is necessary because on a non-separable Hilbert space every operator is reducible.) An operator A on $\mathscr{H}$ is said to be *strongly irreducible* if XAX^{-1} is irreducible for every invertible operator X in $\mathscr{L}(\mathscr{H})$, introduced by F. Gilfeather [6]. Equivalently, an operator A is strongly irreducible if and only if $\{A\}'$ contains no idempotents other than 0 and I.

In [15], strongly irreducible operators are considered as analogues of Jordan blocks on complex separable Hilbert spaces. In the past two decades, D. A. Herrero, Yasuo Watatani, Zejian Jiang, C. K. Fong, Chunlan Jiang, Zongyao Wang, Peiyuan Wu, Youqing Ji, Junsheng Fang and many other mathematicians did a lot of work around this class of operators [8, 9, 10, 11, 12, 13, 18, 19, 20].

On a finite dimensional Hilbert space $\mathscr{K}$, the Jordan canonical form theorem shows that every operator can be uniquely written as a (Banach) direct sum of Jordan blocks up to similarity. This means that for an operator B on $\mathscr{K}$, there is a bounded maximal abelian set of idempotents $\mathscr{Q}$ in $\{B\}'$ and $\mathscr{Q}$ is unique up to similarity in $\{B\}'$. However, to represent certain operators on $\mathscr{H}$, direct sums of Jordan blocks need to be replaced by direct integrals of strongly irreducible operators with regular Borel measures. In [14], we proved that an operator A on $\mathscr{H}$ is similar to a direct integral of strongly irreducible operators if and only if its commutant $\{A\}'$ contains a bounded maximal abelian set of idempotents. Related concepts about direct integrals can be found in [3, 4].

For an operator A on $\mathscr{H}$, A direct integral decomposition of A is said to be a *strongly irreducible decomposition (S. I. D.)* of A if the integrand is strongly

irreducible almost everywhere on the domain of integration. An S. I. D. of A is said to be *unique up to similarity* if for bounded maximal abelian sets of idempotents $\mathscr{P}$ and $\mathscr{Q}$ in $\{A\}'$, there is an invertible operator X in $\{A\}'$ such that $X\mathscr{P}X^{-1} = \mathscr{Q}$.

In this paper, we study when A has unique S. I. D. up to similarity, for A similar to a direct integral of strongly irreducible operators.

2. Upper triangular representation and main theorems

If an operator A in $\mathscr{L}(\mathscr{H})$ is similar to a direct integral of strongly irreducible operators, then there is an invertible operator X in $\mathscr{L}(\mathscr{H})$ such that XAX^{-1} has an S. I. D. in the form

$$XAX^{-1} = (\bigoplus_{n=1}^{\infty} \int_{\Lambda_n} (XAX^{-1})(\lambda)d\mu(\lambda)) \bigoplus \int_{\Lambda_\infty} (XAX^{-1})(\lambda)d\mu(\lambda). \qquad (1)$$

Here μ is a regular Borel measure. Write Λ for $\bigcup_{n=1}^{n=\infty} \Lambda_n$. The sets Λ_∞ and Λ_n for n in $\mathbb{N}$ are bounded Borel and pairwise disjoint. For n in $\mathbb{N}$ and almost every λ in Λ_n, the dimension of the fibre space $\mathscr{H}_\lambda$ is n. For almost every λ in Λ_∞, the dimension of the fibre space $\mathscr{H}_\lambda$ is ∞. (For fibre space, see [1], §2.) Some Λ_ns and Λ_∞ may be of measure zero. The partitioned measure space corresponding to the S. I. D. of XAX^{-1} is denoted by $\{\Lambda, \mu, \{\Lambda_n\}_{n=1}^{n=\infty}\}$.

For a nonzero normal operator N on $\mathscr{H}$, the tensor product $I \otimes N$ does not have unique S. I. D. up to similarity, where I is the identity operator on $\mathscr{H}$ and $\dim\mathscr{H} = \infty$. If A is similar to a normal operator N, then the S. I. D. of A is unique up to similarity if and only if the multiplicity function m_N for N is finite a. e. on $\sigma(N)$. Based on this, we can construct a non-normal operator which does not have unique S. I. D. up to similarity, if $\mu(\Lambda_\infty) \neq 0$ in (1). If $\mu(\Lambda_\infty) = 0$ then the S. I. D. of XAX^{-1} is of the form

$$XAX^{-1} = \bigoplus_{n=1}^{\infty} \int_{\Lambda_n} (XAX^{-1})(\lambda)d\mu(\lambda). \qquad (2)$$

By ([2], Corollary 2), there is a unitary operator U such that the equation

$$U(XAX^{-1})U^*(\lambda) = U(\lambda)(XAX^{-1})(\lambda)U^*(\lambda)$$

holds a. e. on Λ and $U(XAX^{-1})U^*(\lambda)$ is upper triangular in $M_n(\mathbb{C})$ for λ a. e. in Λ_n. Write μ_n for $\mu|_{\Lambda_n}, 1 \leq n < \infty$. Without loss of generality, we assume that

$$XAX^{-1} = \bigoplus_{n=1}^{\infty} \int_{\Lambda_n} \begin{pmatrix} M_{\phi_n} & M_{\phi_{12}^n} & M_{\phi_{13}^n} & \cdots & M_{\phi_{1n}^n} \\ 0 & M_{\phi_n} & M_{\phi_{23}^n} & \cdots & M_{\phi_{2n}^n} \\ 0 & 0 & M_{\phi_n} & \cdots & M_{\phi_{3n}^n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & M_{\phi_n} \end{pmatrix}_{n \times n} (\lambda)d\mu(\lambda), \qquad (3)$$

where $\phi_n, \phi_{ij}^n \in L^\infty(\mu_n)$, M_{ϕ_n} and $M_{\phi_{ij}^n}$ are multiplication operators. The scalar-valued spectral measure for M_{ϕ_n} is $\nu_n \equiv \mu_n \circ \phi_n^{-1}$. Let the set $\{\Gamma_{nm}\}_{m=1}^{m=\infty}$ be a Borel partition of $\sigma(M_{\phi_n})$ corresponding to the multiplicity function m_{ϕ_n} for M_{ϕ_n} on $\sigma(M_{\phi_n})$, where $m_{\phi_n}(\lambda) = m, \forall \lambda \in \Gamma_{nm}$. Write ν_{nm} for $\nu_n|_{\Gamma_{nm}}, 1 \leq m \leq \infty$. We find that the functions m_{ϕ_n} play a significant role on studying the uniqueness of S. I. D. of A up to similarity. (Note that ϕ_{ij}^n does not stand for ϕ_{ij} to the power of n here. The symbol n is only a superscript.)

Definition 2.1. The function m_{ϕ_n} on $\sigma(M_{\phi_n})$ is said to be *the characteristic function of S. I. D.* of XAX^{-1} corresponding to Λ_n.

In the rest of this paper, we write the partitioned measure space corresponding to the S. I. D. of XAX^{-1} as $\{\Lambda, \mu, \{\Lambda_n, m_{\phi_n}\}_{n=1}^{\infty}\}$.

The purpose of this paper is to give a sufficient condition such that the S. I. D. of A in (2) is unique up to similarity. Precisely we prove the following theorems.

Theorem 2.2. *For a fixed n, assume that $T \in \mathscr{L}(\mathscr{H})$ is a direct integral of upper triangular strongly irreducible operators and the corresponding measure space is $\{\Lambda_n, \mu_n, \{\Lambda_n, m_{\phi_n}\}\}$ as in (3). If there is a unitary operator U such that both*

$$UM_{\phi_n}U^* = M_{z_\infty}^{(\infty)} \oplus M_{z_1} \oplus M_{z_2}^{(2)} \oplus \cdots$$

and

$$UM_{\phi_{ij}^n}U^* = M_{\psi_\infty^{n,ij}}^{(\infty)} \oplus M_{\psi_1^{n,ij}} \oplus M_{\psi_2^{n,ij}}^{(2)} \oplus \cdots$$

hold, where $\psi_m^{n,ij}$ and z_m are in $L^\infty(\nu_{nm})$, and $z_m(t) = t, \forall t \in \Gamma_{nm}$. then the following statements are equivalent:

1. *The bounded ν_n-measurable multiplicity function m_{ϕ_n} is simple on $\sigma(M_{\phi_n})$.*
2. *The S. I. D. of T is unique up to similarity.*
3. *The K_0 group of $\{T\}'$ is of the form*

$$K_0(\{T\}') \cong \{f : \sigma(M_{\phi_n}) \to \mathbb{Z} | f \text{ is bounded Borel.}\}.$$

The condition in this theorem is significant and reasonable. We show this in the proofs in §3. The following theorem is a generalized version of the above theorem.

Theorem 2.3. *Assume that $T \in \mathscr{L}(\mathscr{H})$ is a direct integral of upper triangular strongly irreducible operators and the corresponding partitioned measure space is $\{\Lambda, \mu, \{\Lambda_n, m_{\phi_n}\}_{n=1}^{\infty}\}$ as in (3). The set Λ_n is of μ-measure 0 for all but finitely many n in $\mathbb{N}$. If the spectral measures for $\{M_{\phi_n}\}_{n=1}^{\infty}$ are mutually singular and there is a unitary operator $U \in \{T\}'$ satisfying the condition in Theorem 2.2 w. r. t. ϕ_n and ϕ_{ij}^n $(i > j)$ on every Λ_n, then the following statements are equivalent:*

1. *The bounded ν_n-measurable multiplicity function m_{ϕ_n} is simple on $\sigma(M_{\phi_n})$ for n in $\mathbb{N}$.*
2. *The S. I. D. of T is unique up to similarity.*
3. *The K_0 group of $\{T\}'$ is of the form*

$$K_0(\{T\}') \cong \{f : \bigcup_{n=1}^{\infty} \sigma(M_{\phi_n}) \to \mathbb{Z} | f \text{ is bounded Borel.}\}.$$

The rest of this paper is organized as follows. In Section 3, first we prove a special case of Theorem 3.3 in Lemma 3.2 and then we prove Theorem 3.3 in three lemmas. Corollary 3.7 is to characterize the K_0 group of the commutant of the operator T in Theorem 3.3. In Example 3.8, we construct an operator and compute the corresponding K_0 group. Finally, we prove Theorem 2.2 and Theorem 2.3. The operator T in Theorem 3.3 indicates why we add a condition about the unitary operator in Theorem 2.2.

3. Proofs

The following lemma reveals an important property which is applied in other lemmas in this paper.

Lemma 3.1. *Assume that ϕ, $\phi_{ij} \in L^\infty(\mu_n)$, where $1 \le i$, $j \le n$ and Λ_n is as in (1). Multiplication operators M_ϕ and $M_{\phi_{ij}}$ are on $L^2(\mu_n)$. Then the following upper triangular form*

$$T = \begin{pmatrix} M_\phi & M_{\phi_{12}} & M_{\phi_{13}} & \cdots & M_{\phi_{1n}} \\ 0 & M_\phi & M_{\phi_{23}} & \cdots & M_{\phi_{2n}} \\ 0 & 0 & M_\phi & \cdots & M_{\phi_{3n}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & M_\phi \end{pmatrix}_{n \times n} \tag{4}$$

is strongly irreducible a. e. on Λ_n if and only if $\phi_{i,i+1}(\lambda) \ne 0$ holds a. e. on Λ_n for $i = 1, 2, \ldots, n-1$. The underlying Hilbert space of T is denoted by $\mathscr{H}$.

Proof. For λ in Λ_n, if $T(\lambda)$ is strongly irreducible in $M_n(\mathbb{C})$, then there is an invertible operator X in $M_n(\mathbb{C})$ such that $T(\lambda)X = XJ_{\sigma(T(\lambda))}$, where $J_{\sigma(T(\lambda))}$ is a Jordan block in $M_n(\mathbb{C})$ with spectrum $\sigma(T(\lambda))$. The equation $T(\lambda)X = XJ_{\sigma(T(\lambda))}$ becomes

$$\begin{pmatrix} \alpha & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1n} \\ 0 & \alpha & \alpha_{23} & \cdots & \alpha_{2n} \\ 0 & 0 & \alpha & \cdots & \alpha_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \cdots & x_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \cdots & x_{nn} \end{pmatrix} \begin{pmatrix} \alpha & 1 & 0 & \cdots & 0 \\ 0 & \alpha & 1 & \cdots & 0 \\ 0 & 0 & \alpha & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha \end{pmatrix},$$

where $\alpha = \sigma(T(\lambda))$. This equation implies that $x_{ij} = 0$ for $i > j$ and $\alpha_{i-1,i}x_{ii} = x_{i-1,i-1}$ for $i = 2, 3, \ldots, n$. Therefore the invertibility of X implies that $\alpha_{i-1,i} \ne 0$ holds for $i = 2, 3, \ldots, n$.

On the other hand, if $\alpha_{i-1,i} \ne 0$ holds for $i = 2, 3, \ldots, n$, then every operator X in $M_n(\mathbb{C})$ satisfying $T(\lambda)X = XT(\lambda)$ can be expressed in the form

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ 0 & x_{11} & x_{23} & \cdots & x_{2n} \\ 0 & 0 & x_{11} & \cdots & x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_{11} \end{pmatrix}.$$

If X is an idempotent, then it must be I or 0. Thus $T(\lambda)$ is strongly irreducible. $\square$

Lemma 3.2. *Suppose that an operator T is assumed as in (4), $\phi_{i,i+1}(\lambda) \ne 0$ holds a. e. on Λ_n for $i = 1, 2, \ldots, n-1$, and ϕ is one to one a. e. on Λ_n. Then the S. I. D. of T is unique up to similarity.*

Proof. Let $E(\cdot)$ be the spectral measure of M_ϕ. Thus $E^{(n)}(\cdot)$ is the spectral measure of $M_\phi^{(n)}$. Meanwhile, the spectral measures of M_ϕ and M_z have the same range. First we show $\{T\}' \subseteq \{M_\phi^{(n)}\}'$. Equivalently, we need to prove that for every Borel subset σ of $\sigma(M_\phi)$ and $X \in \{T\}'$, the projection $E^{(n)}(\sigma)$ reduces X.

If we write μ_{n1} for $\mu_n|_{\phi^{-1}(\sigma)}$ and μ_{n2} for $\mu_n|_{\Lambda_n \backslash \phi^{-1}(\sigma)}$, then we have

$$[\mathrm{ran}(E^{(n)}(\sigma))] = (L^2(\mu_{n1}))^{(n)}, \ [\mathrm{ran}(I - E^{(n)}(\sigma))] = (L^2(\mu_{n2}))^{(n)}.$$

The operators T and X can be expressed in the form

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \begin{matrix} (L^2(\mu_{n1}))^{(n)} \\ (L^2(\mu_{n2}))^{(n)} \end{matrix} \ , \quad X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \begin{matrix} (L^2(\mu_{n1}))^{(n)} \\ (L^2(\mu_{n2}))^{(n)} \end{matrix} \ ,$$

where

$$T_i = \begin{pmatrix} M_\phi^i & M_{\phi_{12}}^i & M_{\phi_{13}}^i & \cdots & M_{\phi_{1n}}^i \\ 0 & M_\phi^i & M_{\phi_{23}}^i & \cdots & M_{\phi_{2n}}^i \\ 0 & 0 & M_\phi^i & \cdots & M_{\phi_{3n}}^i \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & M_\phi^i \end{pmatrix}_{n \times n} \begin{matrix} L^2(\mu_{ni}) \\ L^2(\mu_{ni}) \\ L^2(\mu_{ni}) \\ \vdots \\ L^2(\mu_{ni}) \end{matrix} \ , \quad i = 1, 2.$$

The equation $T_1 X_{12} = X_{12} T_2$ and the fact that M_ϕ^1 and M_ϕ^2 have mutually singular scalar-valued spectral measures imply that $X_{12} = 0$. In the same way we obtain $X_{21} = 0$. Therefore $X \in \{M_\phi^{(n)}\}'$.

By Lemma 3.1, we compute the equation $TX = XT$ and obtain that the operator X has the form

$$X = \begin{pmatrix} M_\psi & M_{\psi_{12}} & M_{\psi_{13}} & \cdots & M_{\psi_{1n}} \\ 0 & M_\psi & M_{\psi_{23}} & \cdots & M_{\psi_{2n}} \\ 0 & 0 & M_\psi & \cdots & M_{\psi_{3n}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & M_\psi \end{pmatrix}_{n \times n} , \tag{5}$$

where ψ, $\psi_{ij} \in L^\infty(\mu_n)$. Hence every idempotent in $\{T\}'$ is a spectral projection of $M_\phi^{(n)}$. This means that in the commutant of T, there is one and only one bounded maximal abelian set of idempotents. $\qquad\square$

Theorem 3.3. *If an operator T is assumed as in Lemma 3.2 and m is a positive integer, then the S. I. D. of $T^{(m)}$ is unique up to similarity.*

We denote by $\mathscr{P}$ the set of projections in $\{T\}'$. This is the only maximal abelian set of idempotents in $\{T\}'$. The set $\mathscr{P} \oplus \cdots \oplus \mathscr{P}(m$ copies$)$ is a bounded maximal abelian set of idempotents in $M_m(\{T\}')$. We prove Theorem 3.3 in three lemmas.

Lemma 3.4. *If $Q \in M_m(\{T\}')$ is an idempotent, then there is an invertible operator $X \in M_m(\{T\}')$ such that the operator XQX^{-1} belongs to the set $\mathscr{P} \oplus \cdots \oplus \mathscr{P}(m$ copies$)$.*

Proof. The idempotent Q is decomposable with respect to the diagonal algebra generated by the set $\mathscr{P}^{(m)} \equiv \{P \oplus P \oplus \cdots \oplus P(m$ copies$) : P \in \mathscr{P}\}$. The measure

space is Λ_n. Based on (5) in Lemma 3.2, the operator Q can be expressed in the form

$$Q = \begin{pmatrix} M_{\psi^{11}} & \cdots & M_{\psi^{11}_{1n}} & & M_{\psi^{1m}} & \cdots & M_{\psi^{1m}_{1n}} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & M_{\psi^{11}} & & 0 & \cdots & M_{\psi^{1m}} \\ & \vdots & & \ddots & & \vdots & \\ M_{\psi^{m1}} & \cdots & M_{\psi^{m1}_{1n}} & & M_{\psi^{mm}} & \cdots & M_{\psi^{mm}_{1n}} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & M_{\psi^{m1}} & & 0 & \cdots & M_{\psi^{mm}} \end{pmatrix}_{mn \times mn}.$$

There is a unitary operator U_1 such that the operator $Q_1 = U_1 Q U_1^*$ is represented as a block upper triangular operator-valued matrix in the form

$$Q_1 = \begin{pmatrix} Q^1_{11} & Q^1_{12} & Q^1_{13} & \cdots & Q^1_{1n} \\ 0 & Q^1_{11} & Q^1_{23} & \cdots & Q^1_{2n} \\ 0 & 0 & Q^1_{11} & \cdots & Q^1_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & Q^1_{11} \end{pmatrix}_{n \times n},$$

where

$$Q^1_{11} = \begin{pmatrix} M_{\psi^{11}} & \cdots & M_{\psi^{1m}} \\ \vdots & \ddots & \vdots \\ M_{\psi^{m1}} & \cdots & M_{\psi^{mm}} \end{pmatrix}_{m \times m}, \quad Q^1_{ij} = \begin{pmatrix} M_{\psi^{11}_{ij}} & \cdots & M_{\psi^{1m}_{ij}} \\ \vdots & \ddots & \vdots \\ M_{\psi^{m1}_{ij}} & \cdots & M_{\psi^{mm}_{ij}} \end{pmatrix}_{m \times m},$$

for $1 \leq i < j \leq n$. Notice that Q^1_{11} is an idempotent.

Next, we prove that there is an invertible operator X_2 in $M_m(L^\infty(\mu_n))$ such that $X_2 Q^1_{11} X_2^{-1}$ is a projection in the form

$$X_2 Q^1_{11} X_2^{-1} = \begin{pmatrix} M_{\chi_{S_1}} & 0 & \cdots & 0 \\ 0 & M_{\chi_{S_2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{\chi_{S_m}} \end{pmatrix}_{m \times m},$$

where S_i is a Borel subset of Λ_n for $i = 1, 2, \ldots, m$.

For any positive integer k, there is a positive integer l_k such that given any idempotent P in $M_m(\mathbb{C})$ with norm less than k there is an invertible operator X with norm less than l_k such that XPX^{-1} is similar to the corresponding Jordan block. That is because any idempotent in $M_m(\mathbb{C})$ is unitarily equivalent to a block matrix in the form

$$\begin{pmatrix} I & R \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} I & R \\ 0 & I \end{pmatrix} \begin{pmatrix} I & R \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & -R \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}. \tag{6}$$

For the set defined in ([2], Corollary 3)

$$\mathscr{E}_{l_k} = \{(A, J, X) \in M_m(\mathbb{C}) \times M_m(\mathbb{C}) \times M_m(\mathbb{C}) : J \text{ is in Jordan form,} \\ ||X|| \leq l_k, \ ||X^{-1}|| \leq l_k \text{ and } XAX^{-1} = J\},$$

the set $\pi_1(\mathscr{E}_{l_k})$ contains every idempotent whose norm is less than k. By ([2], Theorem 1), the Borel map $\phi_{l_k} : \pi_1(\mathscr{E}_{l_k}) \to \pi_3(\mathscr{E}_{l_k})$ is bounded. Therefore the equivalent class of $\phi_{l_{\lceil \|Q_{11}^1\| \rceil}} \circ Q_{11}^1(\cdot)$ is the X_2 we need in $M_m(L^\infty(\mu_n))$.

Write Q_2 for $(X_2^{(n)})Q_1(X_2^{(n)})^{-1}$. There is a unitary operator U_3 in $M_m(L^\infty(\mu_n))$ such that $(U_3^{(n)})Q_2(U_3^{(n)})^*$ can be expressed in the form

$$(U_3^{(n)})Q_2(U_3^{(n)})^* = \begin{pmatrix} Q_{11}^3 & Q_{12}^3 & Q_{13}^3 & \cdots & Q_{1n}^3 \\ 0 & Q_{11}^3 & Q_{23}^3 & \cdots & Q_{2n}^3 \\ 0 & 0 & Q_{11}^3 & \cdots & Q_{3n}^3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & Q_{11}^3 \end{pmatrix}_{n \times n},$$

where

$$Q_{11}^3 = \begin{pmatrix} M_{\chi_{S_1}} & 0 & \cdots & 0 \\ 0 & M_{\chi_{S_2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{\chi_{S_m}} \end{pmatrix}_{m \times m}.$$

The set S_i is a Borel subset of Λ_n for $i = 1, 2, \ldots, m$, and $S_{i+1} \subseteq S_i$ for $i = 1, 2, \ldots, m-1$. Write Q_3 for $(U_3^{(n)})Q_2(U_3^{(n)})^*$. Notice that Q_3 belongs to $\{U_1 T^{(m)} U_1^*\}'$.

Next, we prove that there is an invertible operator X_4 in $\{U_1 T^{(m)} U_1^*\}'$ such that $X_4 Q_3 X_4^{-1}$ equals the following projection

$$Q_4 = \begin{pmatrix} Q_{11}^3 & 0 & 0 & \cdots & 0 \\ 0 & Q_{11}^3 & 0 & \cdots & 0 \\ 0 & 0 & Q_{11}^3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & Q_{11}^3 \end{pmatrix}_{n \times n}.$$

First, multiply each entry in the lower triangular $m \times m$ matrix form of $Q_{i,i+1}^3$ by -1 and denote this new $m \times m$ matrix form by $X_{i,i+1}^3$, for $i = 1, 2, \ldots, n-1$. In $\{U_1 T^{(m)} U_1^*\}'$, we can construct an operator X_1^3 in the form

$$X_1^3 = \begin{pmatrix} I & X_{12}^3 & *_{13}^1 & \cdots & *_{1n}^1 \\ 0 & I & X_{23}^3 & \cdots & *_{2n}^1 \\ 0 & 0 & I & \cdots & *_{3n}^1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \end{pmatrix}_{n \times n}.$$

The operator X_1^3 is invertible and $\sigma(X_1^3) = \{1\}$. The fact that $\{U_1 T^{(m)} U_1^*\}'$ is a subalgebra of $\mathscr{L}(\mathscr{H}^{(m)})$ implies that the equation $\sigma(X_1^3) = \{1\}$ holds in $\{U_1 T^{(m)} U_1^*\}'$. Thus X_1^3 is invertible in $\{U_1 T^{(m)} U_1^*\}'$. Therefore the operator $(X_1^3)Q_3(X_1^3)^{-1}$ is in the form

$$\begin{pmatrix} Q_{11}^3 & 0 & *_{13}^2 & \cdots & *_{1n}^2 \\ 0 & Q_{11}^3 & 0 & \cdots & *_{2n}^2 \\ 0 & 0 & Q_{11}^3 & \cdots & *_{3n}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & Q_{11}^3 \end{pmatrix}_{n \times n}.$$

Repeat the above procedure. We construct invertible operators X_i^3 one by one in $\{U_1 T^{(m)} U_1^*\}'$, for $i = 1, \ldots, n-1$. After $n-1$ steps, we obtain Q_4. Denote by X_4 the product of X_i^3s. The equation $Q_4 = X_4 Q_3 X_4^{-1}$ holds. Therefore $X = U_1^* X_4 U_3^{(n)} X_2^{(n)} U_1$ is the invertible operator in $M_m(\{T\}')$ such that XQX^{-1} is a projection in $\mathscr{P} \oplus \cdots \oplus \mathscr{P} (m$ copies). $\qquad\square$

Lemma 3.5. *If $\mathscr{Q}$ is a bounded maximal abelian set of idempotents in $M_m(\{T\}')$, then there is a subset $\{Q_i\}_{i=1}^{2^m} \subseteq \mathscr{Q}$ such that for λ in Λ_n, the equation $\{Q_i(\lambda)\}_{i=1}^{2^m} = \mathscr{Q}(\lambda)$ holds a. e. on Λ_n.*

Proof. By the above lemma, we know that for every Q in $\mathscr{Q}$, there is an invertible operator X in $M_m(\{T\}')$ such that XQX^{-1} is a projection in $\mathscr{P} \oplus \cdots \oplus \mathscr{P} (m$ copies). Thus we define a function

$$r_Q(\lambda) = \frac{1}{n} \text{rank}(Q(\lambda)), \ \forall Q \in \mathscr{Q}, \ \lambda \in \Lambda_n.$$

The function r_Q is in the equivalent class of certain simple function. To prove this lemma, we only need to show that there are m idempotents Q_i in $\mathscr{Q}$ such that the equation $r_{Q_i}(\lambda) = 1$, for $i = 1, \ldots, m$ holds a. e. on Λ_n and $Q_i Q_j = 0$, for $i \neq j$. We prove this in two steps.

Step 1, we prove that there is an idempotent Q' in $\mathscr{Q}$ such that the relation $0 < r_{Q'}(\lambda) < m$ holds a. e. on Λ_n.

If the relation $\{r_Q(\lambda) : Q \in \mathscr{Q}\} = \{0, m\}$ holds a. e. on Λ_n, then we can construct a strongly measurable operator-valued constant function $Q'(\cdot)$ satisfying the following properties:

(1) $Q'(\cdot)$ is nontrivial a. e. on Λ_n.
(2) The equivalent class Q' of $Q'(\cdot)$ is a projection in $M_m(\{T\}')$ commuting with every idempotent in $\mathscr{Q}$.
(3) Q' does not belong to $\mathscr{Q}$.

This contradicts with the assumption that $\mathscr{Q}$ is a maximal abelian set of idempotents. Therefore, there are an idempotent Q_1' in $\mathscr{Q}$ and a Borel subset Λ_{n1} of Λ_n with nonzero measure such that the relation $0 < r_{Q_1'}(\lambda) < m$ holds a. e. on Λ_{n1}. Thus there are an idempotent Q_2' in $\mathscr{Q}$ and a Borel subset Λ_{n2} of $\Lambda_n \backslash \Lambda_{n1}$ with nonzero measure such that the relation $0 < r_{Q_2'}(\lambda) < m$ holds a. e. on Λ_{n2}. Carry out this procedure and we obtain a subset $\{Q_i'\}_{i=1}^{\infty}$ of $\mathscr{Q}$ and $\Lambda_n = \bigcup_{i=1}^{\infty} \Lambda_{ni}$. Define

$$Q' = \sum_{i=1}^{\infty} [\chi_{\Lambda_{ni}}] \cdot Q_i', \quad \chi_{\Lambda_{ni}}(\lambda) = \begin{cases} I \in M_{mn}(\mathbb{C}), & \lambda \in \Lambda_{ni}; \\ 0 \in M_{mn}(\mathbb{C}), & \lambda \in \Lambda_n \backslash \Lambda_{ni}. \end{cases}$$

The idempotent Q' is what we want in step 1.

Note that $r_{Q'}$ is in the equivalent class of a simple function. We can write Λ_n in the form of a union of disjoint Borel subsets Λ_{ni}' of Λ_n such that the equation $r_{Q'}(\lambda) = i$ holds a. e. on Λ_{ni}'. (Some Λ_{ni}'s may be of measure zero.) Write

$$\mathscr{Q}_i = \{[\chi_{\Lambda_{ni}'}] Q Q' : Q \in \mathscr{Q}\}, \text{ for } i = 1, \ldots, m-1.$$

Step 2, we prove that for a fixed i larger than 1, if the set Λ_{ni}' is not of measure zero, then there is an idempotent $Q'' \in \mathscr{Q}_i$ such that the relation $0 < Q''(\lambda) < i$ holds a. e. on Λ_{ni}'.

Suppose that the relation $\{r_Q(\lambda) : Q \in \mathscr{Q}_i\} = \{0, i\}$ holds a. e. on Λ'_{ni}. By the above lemma we know that there is an invertible operator $X \in M_m(\{T\}')$ such that the equation

$$XQ'X^{-1}(\lambda) = \begin{pmatrix} I & 0 & & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ & \vdots & \ddots & & \vdots \\ 0 & 0 & & I & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{2n \times 2n} \quad \begin{matrix} \mathbb{C}^{(i)} \\ \mathbb{C}^{(m-i)} \\ \vdots \\ \mathbb{C}^{(i)} \\ \mathbb{C}^{(m-i)} \end{matrix}$$

holds a. e. on Λ'_{ni}. Therefore we can construct an idempotent Q'' in $M_m(\{T\}')$ satisfying that

(1) $Q''(\lambda)$ is a proper sub-idempotent of $Q'(\lambda)$ a. e. on Λ'_{ni}.
(2) Q'' commutes with every idempotent in $\mathscr{Q}_i$.
(3) Q'' does not belong to $\mathscr{Q}_i$.

Thus $\mathscr{Q}$ is not a maximal abelian set of idempotents. This is a contradiction. Therefore there are an idempotent $Q''_1 \in \mathscr{Q}_i$ and a Borel subset Λ'_{ni1} of Λ'_{ni} with nonzero measure such that the relation $0 < r_{Q''_1}(\lambda) < i$ holds a. e. on Λ'_{ni1}. Thus there are an idempotent $Q''_2 \in \mathscr{Q}_i$ and a Borel subset Λ'_{ni2} of $\Lambda'_{ni} \backslash \Lambda'_{ni1}$ with nonzero measure such that the relation $0 < r_{Q''_2}(\lambda) < i$ holds a. e. on Λ'_{ni2}. Carry out this procedure and we obtain a subset $\{Q''_k\}_{k=1}^\infty$ of $\mathscr{Q}_i$ and $\Lambda'_{ni} = \bigcup_{k=1}^\infty \Lambda'_{nik}$. Write

$$Q'' = \sum_{k=1}^\infty [\chi_{\Lambda'_{nik}}] \cdot Q''_k.$$

This idempotent Q'' is what we want in step 2.

After finite steps, we obtain that there is an idempotent Q in $\mathscr{Q}$ such that the equation $r_Q(\lambda) = 1$ holds a. e. on Λ_n.

Repeat the above procedure, we can find m idempotents Q_i in $\mathscr{Q}$ such that the equation $r_{Q_i}(\lambda) = 1$, for $i = 1, \ldots, m$ holds a. e. on Λ_n and $Q_i Q_j = 0$, for $i \neq j$. Thus we can obtain 2^m idempotents that we need.

$\square$

Lemma 3.6. *If $\mathscr{Q}$ is a bounded maximal abelian set of idempotents in $M_m(\{T\}')$, then there is an invertible operator $X \in M_m(\{T\}')$ such that*

$$X\mathscr{Q}X^{-1} = \mathscr{P} \oplus \cdots \oplus \mathscr{P}(m \text{ copies}).$$

Proof. By the above lemma, we can find m idempotents Q_i in $\mathscr{Q}$ such that the equation $r_{Q_i}(\lambda) = 1$, for $i = 1, \ldots, m$ holds a. e. on Λ_n and $Q_i Q_j = 0$, for $i \neq j$. For Q_1, there is an invertible operator $X_1 \in M_m(\{T\}')$ such that $X_1 Q_1 X_1^{-1}$ is a projection in $\mathscr{P} \oplus \cdots \oplus \mathscr{P}(m \text{ copies})$. The invertible operator X_1 can be chosen such that $X_1 Q_1 X_1^{-1}$ is in the form

$$X_1 Q_1 X_1^{-1} = \begin{pmatrix} I & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{m \times m}.$$

Thus there is an invertible operator $X_2 \in M_m(\{T\}')$ such that $X_2 X_1 Q_1 X_1^{-1} X_2^{-1}$ and $X_2 X_1 Q_2 X_1^{-1} X_2^{-1}$ are in the form

$$X_2 X_1 Q_1 X_1^{-1} X_2^{-1} = \begin{pmatrix} I & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{m \times m} ,$$

$$X_2 X_1 Q_2 X_1^{-1} X_2^{-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{m \times m} .$$

By this procedure, we obtain $\{X_i\}_{i=1}^m$ such that $X Q_i X^{-1}$ is in the form

$$X Q_i X^{-1} = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & I & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}_{m \times m} \quad \begin{matrix} \vdots \\ i \text{ th} \\ \vdots \end{matrix} ,$$

where $X = X_m X_{m-1} \cdots X_1$. This X in $M_m(\{T\}')$ is the invertible operator that we need.

$\square$

With the above three lemmas, we finish the proof of Theorem 3.3.

Corollary 3.7. If an operator T is assumed as in Theorem 3.3, then the K_0 group of $\{T\}'$ is isomorphic to the set

$$K_0(\{T\}') \cong \{f : \sigma(M_\phi) \to \mathbb{Z} \,|\, f \text{ is bounded Borel.}\}. \tag{7}$$

We give an example to show that the K_0 group of the commutant of an operator T as in the above Corollary is isomorphic to the corresponding set as (7).

Example 3.8. Let $z(t) = t$, $t \in [0, 1]$. The multiplication operator M_z corresponding to z is the operator on $L^2([0, 1])$ defined by

$$(M_z f)(t) = t \cdot f(t), \quad f \in L^2([0, 1]).$$

Denote by T the 2×2 operator-valued matrix in the form

$$T = \begin{pmatrix} M_z & M_\psi \\ 0 & M_z \end{pmatrix}, \quad \psi \in L^\infty([0, 1]), \ \psi(\lambda) \neq 0 \ \text{ a. e. on } [0, 1].$$

By Lemma 3.2, we know that every idempotent P in $\{T\}'$ is of the form

$$P = \begin{pmatrix} M_{\chi_S} & 0 \\ 0 & M_{\chi_S} \end{pmatrix},$$

where S is a Borel subset of $[0,1]$. By Theorem 3.3, we know that, for any positive integer m, in $\{T^{(m)}\}'$ every idempotent is similar to a projection in the form

$$
P = \begin{pmatrix}
M^{(2)}_{\chi_{S_1}} & 0 & \cdots & 0 \\
0 & M^{(2)}_{\chi_{S_2}} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & M^{(2)}_{\chi_{S_m}}
\end{pmatrix}, \tag{8}
$$

where S_i is a Borel subset of $[0,1]$. Denote the standard trace on $M_{2m}(\mathbb{C})$ by Tr. The bounded function $\frac{1}{2}\mathrm{Tr}(P(\lambda))$ maps $[0,1]$ in $\mathbb{N}$ for almost every λ in $[0,1]$. Therefore, in $\bigcup_{m=1}^{\infty}\{T^{(m)}\}'$, for every idempotent P, denote by $[P]$ the class of idempotents similar to P. In every $[P]$, there are projections as (8). Define $\rho([P]) \equiv \frac{1}{2}\mathrm{Tr}(P(\cdot))$. It is easy to prove that the set $\{[P] : P \in \bigcup_{m=1}^{\infty}\{T^{(m)}\}'\}$ is isomorphic to the set $\{f : [0,1] \to \mathbb{N} | f \text{ is bounded Borel.}\}$ and ρ is the isomorphic map. By K-theory for Banach algebras, we obtain

$$
K_0(\{T\}') \cong \{f : [0,1] \to \mathbb{Z} | f \text{ is bounded Borel.}\}.
$$

Lemma 3.9. *Suppose that M_ϕ is the multiplication operator on $L^2(\mu_n)$ as in Lemma 3.2. Then for every idempotent $P \in \{M_\phi^{(\infty)}\}'$, there is an invertible operator X in $\{M_\phi^{(\infty)}\}'$ such that XPX^{-1} is a diagonal projection in $\{M_\phi^{(\infty)}\}'$.*

Proof. The idempotent P is decomposable on $(\Gamma, \mu, \{\Gamma_\infty\})$, $\Gamma = \Lambda_n$. For every λ in Γ, the fiber space $\mathscr{H}_\lambda$ has dimension ∞. First, we construct a bounded Borel set

$$
\{(\lambda, Y, Q) \in \Lambda_n \times \mathscr{L}(\mathscr{H}_\lambda) \times \mathscr{L}(\mathscr{H}_\lambda) : Y^{-1}P(\lambda)Y = Q,
$$
$$
Q \text{ is a diagonal projection}, \|Y\| \le k, \|Y^{-1}\| \le k\},
$$

where k is a positive integer large enough. By ([3], Proposition 2.1), we obtain an invertible operator X in $\{M_\phi^{(\infty)}\}'$ such that XPX^{-1} is a diagonal projection. The operator X is what we need.

$\square$

Lemma 3.10. *If an operator $T \in \mathscr{L}(\mathscr{H})$ is assumed as in Lemma 3.2, then for every idempotent Q in $\{T^{(\infty)}\}'$, there is an invertible operator X in $\{T^{(\infty)}\}'$ such that XQX^{-1} is in $\mathscr{P} \oplus \cdots \oplus \mathscr{P} \oplus \cdots (\infty \text{ copies})$.*

By the proofs of Lemma 3.4 and Lemma 3.9, we obtain this lemma.

Proof of Theorem 2.2. By Theorem 3.3 and Corollary 3.7, we obtain $(1) \Rightarrow (2)$ and $(1) \Rightarrow (3)$. When the multiplicity function m_{ϕ_n} for M_{ϕ_n} takes ∞ on $\sigma(M_{\phi_n})$, we can construct two bounded maximal abelian sets of idempotents in the commutant of T which are not similar to each other in $\{T\}'$. By Lemma 3.10, we know that if m_{ϕ_n} takes ∞ on $\sigma(M_{\phi_n})$, then the K_0 group of $\{T(\lambda)\}'$ is 0 a. e. on $\Gamma_{n\infty}$. $\square$

Proof of Theorem 2.3. Denote by T_n the restriction of T acting on $(L^2(\mu_n))^{(n)}$. The operator T can be expressed as $\bigoplus_{n=1}^{\infty} T_n$. Since the spectral measures of $\{M_{\phi_n}\}_{n=1}^{\infty}$ are mutually singular, we obtain

$$
\{T\}' = \bigoplus_{n=1}^{\infty} \{T_n\}'.
$$

The rest of the proof is essentially an application of Theorem 2.2. $\square$

Acknowledgments

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